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DYNAMIC INSTABILITY OF SHEAR-DEFORMABLE VISCOELASTIC LAMINATED PLATES BY LYAPUNOV EXPONENTS†

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Abstract—The dynamic stability of viscoelastic laminated plates, subjected to a harmonic in-plane excitation, is analyzed. The viscoelastic behavior is caused by the polymeric matrix of the fiber-reinforced material, and a micromechanical analysis provides the time-dependent relaxation functions of the unidirectional lamina. The Boltzmann representation involved in the stress-strain relations of the laminated plate leads to an integro-differential equation of motion, obtained within the first-order shear deformation theory. For this case, a dynamic stability analysis which employs the concept of Lyapunov exponents is performed, and is shown to be very efficient.

INTRODUCTION

Most of the investigations on the buckling of structures that can be found in the literature pertain to the buckling of elastic structures. However, it is well known that many materials exhibit viscoelastic behavior, namely the response is time, as well as history, dependent. There are a number of studies dedicated to the analysis of the buckling of viscoelastic structures. For example, the quasi-static buckling of viscoelastic columns was considered by Flügge (1967) and Glockner and Szyszkowski (1987), whereas the buckling of homogeneous plates was treated by Troyanovskii (1970). Recently, the quasi-static buckling of viscoelastic laminated plates was treated by Cederbaum and Aboudi (1989a).

The analysis of the dynamic stability of a structure, subjected to a periodic loading, is much more complicated due to the existence of inertia terms in the governing equations. It turns out that for certain relationships between the driving frequency and the natural one, dynamic instability occurs, in the sense that the amplitude of the response increases without bound. The problem of the dynamic instability of elastic structures (columns, plates and shells) was investigated by Bolotin (1964), where the instability regions were constructed by using Fourier analysis. Extensive bibliography and further results on this problem were given by Evan-Iwanowski in a review paper (1965) and in a monograph (1976).

The analysis of the dynamic buckling of viscoelastic structures has further complications, since the Boltzmann representation involved in the stress–strain relations leads to an integro-differential equation of motion, rather than an ordinary differential equation as in the elastic case. Chandiramani *et al.* (1989) determined the dynamic stability of orthotropic shear-deformable viscoelastic composite plates subjected to constant in-plane edge loads. Stevens (1966) and Matyash (1973) used approximate methods which involve small

† Dedicated to the 65th anniversary of birth of Professor B. Budiansky.

parameter expansions and string–dashpot representations to analyze this problem, while Szyszkowski and Glockner (1985) with the same representation, analyzed the dynamic instability of a viscoelastic column using the concept of Lyapunov functions. Bogdanovich (1987) analyzed composite shells.

In Aboudi *et al.* (1990), a general approach is offered for the determination of whether a viscoelastic homogeneous plate, subjected to a harmonic in-plane excitation, is stable. The analysis was based on the evaluation of the Lyapunov exponents, according to which the stability (or instability) of the plate was established. The viscoelastic representation is expressed by the Boltzmann superposition principle, which allows the modeling of any linear viscoelastic material. Following Goldhirsch *et al.* (1987), an efficient alogirthm was suggested for the computation of these Lyapunov exponents. Positive values of Lyapunov exponents indicate an instability situation. The accuracy of the stability prediction was checked in the cases of a perfectly elastic plate as well as a viscoelastic plate governed by the Voigt–Kelvin model, for both of which the stability analysis can be independently treated by the method discussed in Bolotin's monograph (1964). It turns out that excellent agreement between the two approaches exists. Then, the analysis of a viscoelastic plate governed by the standard linear solid model was derived and applied to verify that a dynamic stability exists.

In the present paper the approach utilizing the Lyapunov exponents for the determination of the dynamic stability is applied to viscoelastic laminated plates, in which polymeric matrices are employed. It is well known that this type of material exhibits viscoelastic behavior. Previous investigations by Aboudi and Cederbaum (1989), Cederbaum and Aboudi (1989b) and Cederbaum (1990) showed that these viscoelastic effects are significant both from the qualitative and the quantitative points of view.

Since the overall behavior of the single lamina, in the laminated plate, is viscoelastic and transversely isotropic, it is necessary to determine the five independent relaxation (or creep) functions, which characterize its behavior. To this end a micromechanics theory, developed by Aboudi and reviewed in Aboudi (1989), is utilized. This micromechanics theory is based in the analysis of a periodic array of fibers, embedded regularly in the matrix. By studying the detailed interaction of the fibers and the matrix constituents, one can generate the overall properties of the composite. The usefulness of this theory was demonstrated in analyzing the behavior of viscoelastic laminated plates under quasi-static (transverse and in-plane) loads (Cederbaum and Aboudi, 1989a), dynamic loads (Cederbaum and Aboudi, 1989b) and random loads (Cederbaum, 1990).

In the present investigation we deal with the problem of the dynamic instability of antisymmetric angle-ply laminated plates. The equations of motion are derived by using the first-order shear deformation theory (FSDT). The single lamina consists of unidirectional elastic T-300 graphite fibers reinforcing a viscoelastic epoxy matrix (Epon 815 mixed with Versamid 140), which was characterized by Moehlenpha *et al.* (1971). It is shown that the viscoelasticity of the resin matrix reduces the dynamic instability region of the laminated plate as compared with the perfectly elastic plate. This implies that, while for certain load's characteristics the perfectly elastic composite plate is dynamically unstable, the presence of the viscoelasticity might stabilize the plate.

COMPOSITE EFFECTIVE ELASTIC CONSTANTS

Let (x_1, x_2, x_3) denote a Cartesian system of coordinates with x_1 oriented in the fiber direction of a unidirectional fiber-reinforced elastic composite. The constitutive law for the effective transversely isotropic behavior of such composites can be determined from a micromechanics analysis in the form (Aboudi, 1989)

$$\bar{\sigma} = \mathbf{E}\bar{\varepsilon} \tag{1}$$

where $\bar{\sigma} = (\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{33}, \bar{\sigma}_{12}, \bar{\sigma}_{13}, \bar{\sigma}_{23})$ is the average stress, $\bar{\varepsilon} = (\bar{\varepsilon}_{11}, \bar{\varepsilon}_{22}, \bar{\varepsilon}_{33}, 2\bar{\varepsilon}_{12}, 2\bar{\varepsilon}_{13}, 2\bar{\varepsilon}_{23})$ is the average strain, and

$$\mathbf{E} = \begin{bmatrix} e_{11} & e_{12} & e_{12} & 0 & 0 & 0 \\ & e_{22} & e_{23} & 0 & 0 & 0 \\ & & e_{22} & 0 & 0 & 0 \\ & & & e_{44} & 0 & 0 \\ & & & & e_{44} & 0 \\ \text{symm.} & & & & & \frac{1}{2}(e_{22}-e_{23}) \end{bmatrix} \quad (2)$$

where e_{ij} are the effective elastic constants, whose explicit expressions in terms of the fibers and matrix properties and the reinforcement volume ratio can be found in Aboudi's review paper (1989).

FORMULATION

Within FSDT, the following displacement field across the plate thickness is considered:

$$\begin{aligned} V_1(x, y, z, t) &= U(x, y, t) + z\psi_x(x, y, t) \\ V_2(x, y, z, t) &= V(x, y, t) + z\psi_y(x, y, t) \\ V_3(x, y, t) &= W(x, y, t). \end{aligned} \quad (3)$$

Here V_1, V_2 and V_3 are the time-dependent components of the three-dimensional displacement vector in the x, y and z directions, respectively, while U, V and W denote the displacements of a point (x, y) on the mid-plane $z \equiv 0$ and ψ_x and ψ_y are the rotations of the normals to the mid-plane about the y and x axis, respectively.

The equations of motion of laminated plate subjected to in-plane loads and inertial forces, obtained by using the first-order shear deformation theory, are [see Whitney and Pagano (1970)]

$$\begin{aligned} N_{xx,x} + N_{xy,y} &= I_1\ddot{U} + I_2\ddot{\psi}_x \\ N_{xy,x} + N_{yy,y} &= I_1\ddot{V} + I_2\ddot{\psi}_y \\ Q_{xx,x} + Q_{yy,y} - N_1 W_{,xx} - N_2 W_{,yy} &= I_1\ddot{W} \\ M_{xx,x} + M_{xy,y} - Q_{xx} &= I_3\ddot{\psi}_x + I_2\ddot{U} \\ M_{xy,x} + M_{yy,y} - Q_{yy} &= I_3\ddot{\psi}_y + I_3\ddot{V} \end{aligned} \quad (4)$$

where N_1 and N_2 are in-plane loads in the x and y directions, respectively, the inertia terms are defined as

$$(I_1, I_2, I_3) = \int_{-h/2}^{h/2} \rho(1, z, z^2)\,\mathrm{d}z,$$

while the stress and moment resultants, each per unit length, are respectively given by

$$(N_{xx}, N_{yy}, N_{xy}, Q_{xx}, Q_{yy}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})\,\mathrm{d}z$$

$$(M_{xx}, M_{yy}, M_{xy}) = \int_{-h/2}^{h/2} z(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})\,\mathrm{d}z$$

where σ_{ij} $(i,j \equiv x, y)$ are the stress components.

Using eqn (3) in conjunction with the strain–displacements relations, the following relations for the strain components ε_{ij} are obtained;

$$
\begin{aligned}
\varepsilon_{xx} &= U_{,x} + z\psi_{x,x} \\
\varepsilon_{yy} &= V_{,y} + z\psi_{y,y} \\
\varepsilon_{xy} &= U_{,y} + V_{,x} + z(\psi_{x,y} + \psi_{y,x}) \\
\varepsilon_{xz} &= \psi_x + W_{,x} \\
\varepsilon_{yz} &= \psi_y + W_{,y}.
\end{aligned} \tag{5}
$$

With **E** in (1) representing the five independent elastic constants of the equivalent transversely isotropic material which represent the unidirectional composite, it is possible to obtain the five time-dependent functions which characterize the viscoelastic composite whose phases are viscoelastic materials. Each phase (fiber or matrix) is represented by the Boltzmann's superposition principle (Christensen, 1982), which in tensorial notation is given in the convolution form as follows

$$
\sigma_{ij}^{(p)}(t) = c_{ijkl}^{(p)}(t)\varepsilon_{kl}^{(p)}(0) + \int_{0^+}^{t} c_{ijkl}^{(p)}(t-\tau)\dot{\varepsilon}_{kl}^{(p)}(\tau)\,\mathrm{d}\tau \tag{6}
$$

where $c_{ijkl}^{(p)}$ are the relaxation functions of the p-phase. By employing the micromechanical analysis in conjunction with the correspondence principle in the transform domain, and by inverting the Laplace transform (Cederbaum and Aboudi, 1989a), one obtains the time-dependent relaxation functions of the unidirectional lamina denoted by $c_{ij}(t)$. The constitutive relations of the lamina are of the same form as in eqn (6), but with effective relaxation functions $c_{ij}(t)$ replacing the relaxation functions of the phase.

The constitutive relations of the viscoelastic plate can be written in the form

$$
\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
 & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
 & & A_{66} & B_{16} & B_{26} & B_{66} \\
 & & & D_{11} & D_{12} & D_{16} \\
 & & & & D_{22} & D_{26} \\
\text{symmetric} & & & & & D_{66}
\end{bmatrix} *
\begin{Bmatrix} U_{,x} \\ V_{,y} \\ U_{,y} + V_{,x} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix}
$$

and

$$
\begin{Bmatrix} Q_{yz} \\ Q_{xz} \end{Bmatrix} = k \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} * \begin{Bmatrix} \psi_y + W_{,y} \\ \psi_x + W_{,x} \end{Bmatrix} \tag{7}
$$

where the asterisk denotes the convolution operators. Here k is the shear correction factor and A_{ij}, B_{ij} and D_{ij} are defined for viscoelastic materials as

$$
(A_{ij} \equiv A_{ij}(t), B_{ij} \equiv B_{ij}(t), D_{ij} \equiv D_{ij}(t)) = \int_{-h/2}^{h/2} Q_{ij}(t)(1, z, z^2)\,\mathrm{d}z \quad i, j = 1, 2, 6
$$

$$
A_{ij} \equiv A_{ij}(t) = \int_{-h/2}^{h/2} c_{ij}(t)\,\mathrm{d}z \quad i, j = 4, 5 \tag{8}
$$

and $Q_{ij}(t)$ are the reduced components of the relaxation functions $c_{ij}(t)$.

For the case of a square ($0 \leqslant x, y \leqslant a$) antisymmetric angle-ply laminated plate, simply supported all around, the solution functions for the five unknown functions can be written using the method of separation of variables, as

$$U(x,y,t) = \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a}f_u(t)$$

$$V(x,y,t) = \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{a}f_v(t)$$

$$W(x,y,t) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}f_w(t)$$

$$\psi_x(x,y,t) = \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{a}f_x(t)$$

$$\psi_y(x,y,t) = \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a}f_y(t). \tag{9}$$

Let $\alpha_m \equiv m\pi x/a$ and $\beta_n \equiv n\pi y/a$, then the various terms of the equations of motion (4) are given, for each m and n, by (the subscripts m and n are omitted, for convenience):

$$\begin{aligned}
N_{xx,x} &= \{\alpha^2[A_{11}u(t)]+\alpha\beta[A_{12}v(t)]+\alpha^2[B_{16}y(t)]+\alpha\beta[B_{16}x(t)]\}SC\\
N_{xy,x} &= \{\alpha\beta[A_{66}v(t)]+\beta^2[A_{66}u(t)]+\beta^2[B_{16}x(t)]+\alpha\beta[B_{26}y(t)]\}SC\\
N_{xy,y} &= \{\alpha^2[A_{66}v(t)]+\alpha\beta[A_{66}v(t)]+\alpha^2[B_{16}x(t)]+\alpha\beta[B_{26}y(t)]\}CS\\
N_{yy,y} &= \{\alpha\beta[A_{12}u(t)]+\beta^2[A_{22}v(t)]+\alpha\beta[B_{26}y(t)]+\beta^2[B_{26}x(t)]\}CS\\
Q_{xx,x} &= k\{\alpha^2[A_{55}w(t)]+\alpha[A_{55}x(t)]\}SS\\
Q_{yy,y} &= k\{\beta^2[A_{44}w(t)]+\beta[A_{44}y(t)]\}SS\\
M_{xx,x} &= \{\alpha^2[B_{16}v(t)]+\alpha\beta[B_{16}u(t)]+\alpha^2[D_{11}x(t)]+\alpha\beta[D_{12}y(t)]\}CS\\
M_{yy,y} &= \{\alpha\beta[B_{16}u(t)]+\beta^2[B_{26}v(t)]+\alpha\beta[D_{66}y(t)]+\beta^2[D_{66}x(t)]\}CS\\
M_{xy,x} &= \{\alpha^2[B_{16}u(t)]+\alpha\beta[B_{26}v(t)]+\alpha^2[B_{66}y(t)]+\alpha\beta[D_{66}x(t)]\}SC\\
M_{yy,y} &= \{\alpha\beta[B_{16}v(t)]+\beta^2[B_{26}u(t)]+\alpha\beta[D_{12}x(t)]+\beta^2[D_{22}y(t)]\}SC\\
Q_{xx} &= k\{\alpha[A_{55}w(t)]+[A_{55}x(t)]\}CS\\
Q_{yy} &= k\{\beta[A_{44}w(t)]+[A_{44}y(t)]\}SC
\end{aligned} \tag{10}$$

where

$$SC = \sin\alpha x\cos\beta y;\quad CS \equiv \cos\alpha x\sin\beta y;\quad SS = \sin\alpha x\sin\beta y.$$

In addition,

$$A_{11}u(t) = A_{11}(t)f_u(0)+\int_{0^+}^{t}A_{11}(t-\tau)\dot{f}_u(\tau)\,d\tau,$$

so that in general

$$F_{ij}q(t) = F_{ij}(t)f_q(0)+\int_{0^+}^{t}F_{ij}(t-\tau)\dot{f}_q(\tau)\,d\tau, \tag{11}$$

with $F_{ij} = A_{ij}, B_{ij}, D_{ij}$. The in-plane loading contains constant (N_{xs}) and periodic (N_{xd}) terms, so that

$$N_x = N_{xs} + N_{xd}\cos\theta t$$

$$N_y = N_{ys} + N_{yd}\cos\theta t \qquad (12)$$

where t is the time and θ is the load frequency.

Substituting eqns (10)–(12) into (4) we obtain a set of five integro-differential equations, which govern the motion of the shear deformable viscoelastic laminated plate subjected to in-plane parametric loading.

METHOD OF ANALYSIS

We are interested in the stability of the unperturbed equilibrium of the viscoelastic laminated plate. To this end we investigate the set of the integro-differential equations of the perturbed motion [eqns (4)] to evaluate the corresponding Lyapunov exponents. This procedure was employed by Aboudi *et al.* (1990), where the stability of a homogeneous plate represented by a single integro-differential equation of motion was investigated.

For the treatment of an ordinary differential equation with time-dependent coefficients, Lyapunov introduced the concept of characteristic numbers, the sign of which determines whether the unperturbed motion is stable [see e.g. Hahn (1967)]. The negative values of these characteristic numbers are presently referred to as the Lyapunov exponents. According to Lyapunov, if all the exponents are negative, then the unperturbed motion is asymptotically stable. In addition, Chetaev (1960, 1961) proved that if one of the Lyapunov exponents is positive then the unperturbed motion is unstable.

Recently, Lyapunov exponents became a powerful tool in the study of chaotic motion [see e.g. the recent monograph by Moon (1987)] and for the investigation of nonlinear dynamic systems [see Goldhirsch *et al.* (1987)]. From the above discussion, it follows that it suffices to compute the largest Lyapunov exponent for the determination of the stability or instability of the unperturbed motion of the viscoelastic plate in question. The following procedure due to Goldhirsch *et al.* (1987), provides the largest Lyapunov exponent of the system (Berge *et al.*, 1987).

Consider the system of ordinary differential equations

$$\dot{\mathbf{y}}(t) = \mathbf{G}(t)\mathbf{y}(t) + \mathbf{g}(t) \qquad (13)$$

with initial condition that were normalized such that

$$\|\mathbf{y}(0)\| = 1, \qquad (14)$$

where $\|\cdot\|$ is the Euclidean norm. This system is solved numerically [e.g. by using the high-order Runge–Kutta–Verner procedure] for a chosen time interval T. The resulting solution $\mathbf{y}(T)$ is then normalized to obtain

$$\mathbf{z}(T) = \frac{\mathbf{y}(T)}{\|\mathbf{y}(T)\|}. \qquad (15)$$

In addition, we compute

$$U_1 = \ln\|\mathbf{y}(T)\|. \qquad (16)$$

The same system of equations is solved now for the next time interval $T < t < 2T$ but with the initial values $\mathbf{z}(T)$. The numerical solution will produce $\mathbf{y}(2T)$ and

$$U_2 = \ln \|\mathbf{y}(2T)\|$$

$$\mathbf{z}(2T) = \frac{\mathbf{y}(2T)}{\|\mathbf{y}(2T)\|}$$

and the procedure is continued. After NT intervals we have $U_1, \ldots, U_N$. Let

$$\mu_N = \frac{\sum_{j=1}^{N} U_j}{NT}. \tag{17}$$

It should be mentioned that although it appears that the time interval T can be chosen arbitrarily, the determination of the Lyapunov exponents is obtained asymptotically by taking a sufficient number of time intervals (NT). It is demonstrated by Goldhirsch *et al.* (1987) that μ_N when plotted against $1/NT$ tends to a positive number as $1/NT$ approaches zero, if the system is unstable. The rate of convergence proved to be fast as compared to a standard procedure (Goldhirsch *et al.*, 1987). It should be noted that the implementation of the above procedure in a computer program is straightforward.

In order to reduce the governing equations to a system of first-order equations of the form (13), let $f_q(0)$ and $\dot{f}_q(0)$ be the initial values of $f_q(t)$ and $\dot{f}_q(t)$, respectively, at the first time interval $\Delta t = T/M$, where M is a chosen positive integer. Any relaxation function in eqn (11) can be written in the form (i, j are omitted):

$$F(t) = F_0[1-\psi(t)] \tag{18}$$

where $\psi(t) \equiv 0$ at $t \leqslant 0$. Let

$$H(t) \equiv F(t)f_q(0) + \int_{0^+}^{t} F(t-\tau)\dot{f}_q(\tau)\,\mathrm{d}\tau = F_0\left[f_q(t) - \psi(t)f_q(0) - \int_{0^+}^{t} \psi(t-\tau)\dot{f}_q(\tau)\,\mathrm{d}\tau\right],$$

$$0 \leqslant t \leqslant \Delta t \tag{19}$$

and for those cases where $\psi(t-\tau) = \psi_1(t)\psi_2(\tau)$, $H(t)$ can be approximated in the form

$$H(t) = F_0[f_q(t) - \psi(t)f_q(0) - \dot{f}_q(t)\psi_3(t)] \tag{20}$$

where

$$\psi_3(t) = \psi_1(t)\int_{0^+}^{t} \psi_2(\tau)\,\mathrm{d}\tau.$$

This representation is given to each time-dependent term in eqns (10), where the function $\psi(t)$, for each case, is derived by using the micromechanics theory. In the second time interval $\Delta t \leqslant t \leqslant 2\Delta t$, with $f_q(0)$ and $\dot{f}_q(0)$ denoting the initial values of $f(t)$ and $\dot{f}(t)$ at $t = \Delta t$, we again use eqn (18) by shifting the time reference. Consequently, for any $(p-1)\Delta t \leqslant t \leqslant p\Delta t$, $p = 1, 2, 3, \ldots$, the equations of motion (4) can be written in the form of eqn (13), where the components of the (1×10) vector $\mathbf{g}(t)$ and the (10×10) matrix $\mathbf{G}$ can be determined from eqns (4) in conjunction with eqn (10).

APPLICATIONS

In the following examples, square angle-ply laminated plates simply-supported all around are investigated. A four-layered [45°, −45°, 45°, −45°] plate is considered, where the total thickness is 0.5 mm, and the length-to-thickness ratio is fixed at 20.

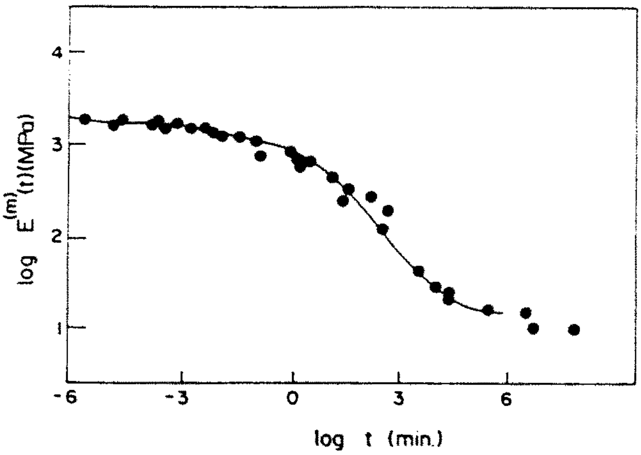

Fig. 1. Measured data (Moehlenpha et al. 1971) of an epoxy matrix at 50 C. The solid line is its representation by cubic splines.

Results are given for a viscoelastic epoxy matrix (Epon 815 mixed with Versamid 140) whose measured time-dependent Young's modulus $E^{(m)}(t)$ was reported in Moehlenpha *et al.* (1971) at 50°C. Cubic splines were fitted to these measured data in the region $0 \leqslant t \leqslant 10^6$ min. The agreement between the measured values and the cubic splines representation, as shown in Fig. 1, is excellent. The Poisson's ratio of the matrix was assumed to be constant, $\nu_m = 0.38$ (Schapery, 1974). It should be noted, however, that the micromechanics analysis (Aboudi, 1989) allows the use of time-dependent Poisson's ratios.

The T-300 graphite elastic fibers were taken to reinforce the epoxy matrix with a 60% volume ratio. The elastic constants of the transversely isotropic graphite fibers are $E_L^{(f)} = 220$ GPa, $\nu_L^{(f)} = 0.3$, $E_T^{(f)} = 22$ GPa, $\nu_T^{(f)} = 0.35$ and $G_L^{(f)} = 22$ GPa, where $E_L^{(f)}$ and $\nu_L^{(f)}$ denote the longitudinal Young's modulus and Poisson's ratio, $E_T^{(f)}$ and $\nu_T^{(f)}$ are the tranverse Young's modulus and Poisson's ratio and $G_L^{(f)}$ is the axial shear modulus.

In order to obtain the relaxation function $F(t)$ in the form of eqn (18), let us assume that $F(t)$ can be described as

$$F(t) = A + B\,e^{-\gamma t} = (A+B)\left[1 - \frac{B}{A+B}(1 - e^{-\gamma t})\right] = F_0[1 - \psi(t)] \tag{21}$$

where

$$F_0 = A + B \quad \text{and} \quad \psi(t) = \frac{B}{A+B}(1 - e^{-\gamma t}).$$

This yields

$$F(t-\tau) = F_0[1 - \psi(t-\tau)] = (A+B)\left[1 - \frac{B}{A+B}(1 - e^{-\gamma t}\,e^{\gamma\tau})\right]. \tag{22}$$

With eqns (21), (22), the function $H(t)$ [eqn (20)] becomes

$$H(t) = F_0\left[f_q(t) - \frac{B}{A+B}f_q(0) - \frac{B}{A+B}\int_{0^+}^{t}\dot{f}_q(\tau)\,d\tau + \frac{B}{A+B}e^{-\gamma t}\int_{0^+}^{t}e^{\gamma\tau}\dot{f}_q(\tau)\,d\tau\right]$$

$$= F_0\left[f_q(t) - \frac{B}{A+B}f_q(t) - \frac{B}{A+B}e^{-\gamma t}\int_{0^+}^{t}e^{\gamma\tau}\dot{f}_q(\tau)\,d\tau\right]. \tag{23}$$

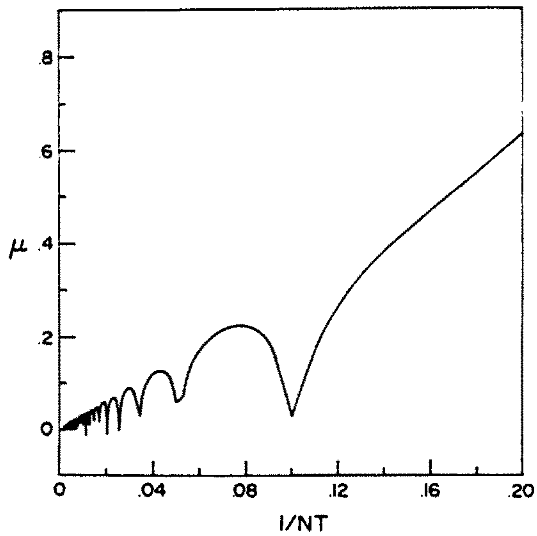

Fig. 2. Lyapunov exponents for an elastic plate, $\eta = 0.0$.

Within each Δt and the appropriate initial conditions, $f_q(0)$ and $\dot{f}_q(0)$, $H(t)$ is approximated as

$$H(t) \simeq F_0\left[\left(1-\frac{B}{A+B}\right)f_q(t)-\frac{B}{A+B}e^{-\gamma t}f_q(0)-\frac{B}{A+B}(1-e^{-\gamma\Delta t})/\gamma\dot{f}_q(t)\right]$$

$$= Af_q(t)-Be^{-\gamma t}f_q(0)-B\frac{1-e^{-\gamma\Delta t}}{\gamma}\dot{f}_q(t). \quad (24)$$

The five time-dependent relaxation functions of the lamina obtained within the micromechanics theory are (moduli in Giga Pascals and time minutes):

$$E_L = 132.0+0.70\exp(-0.0016t)$$
$$E_T = 0.050+6.29\exp(-0.0018t)$$
$$G_{LT} = 0.013+2.14\exp(-0.0020t)$$
$$G_{TT} = 0.016+2.05\exp(-0.0018t)$$
$$\nu_{LT} = 0.34+0.0008\exp(-0.0017t) \quad (25)$$

where E_L and E_T are the longitudinal and transverse relaxation functions, G_{LT} and G_T are the corresponding shear relaxation functions and ν_{LT} is the major time-dependent Poisson ratio. These functions enable one to determine the time-dependent functions $A_{ij}(t)$, $B_{ij}(t)$ and $D_{ij}(t)$.

The in-plane load considered is in the y direction only ($N_{dx} = 0$) and where $N_{ys} = 0$. The load frequenncy, θ, is taken as twice the fundamental natural frequency of the elastic laminated plate, derived from the free vibration problem [see e.g. Cederbaum *et al.* (1989)]. The amplitude of the in-plane force is given by $N_{yd} = \eta 2N$, where N is the static buckling load [see e.g. Srinivas and Rao (1970)] and η is a coefficient.

The following cases were considered: (i) elastic plate [$\psi(t) = 0$ in all stiffnesses] with $\eta = 0.0$—Fig. 2, (ii) viscoelastic plate with $\eta = 0.0$—Fig. 3 and with $\eta = 0.4$—Fig. 4.

In the elastic case the Lyapunov exponents are approaching zero, indicating stability of a system without damping. In the viscoelastic counterpart the exponents are approaching negative values implying that the stability exists. When the system is unstable, the Lyapunov exponents are approaching positive values, both for elastic and viscoelastic cases.

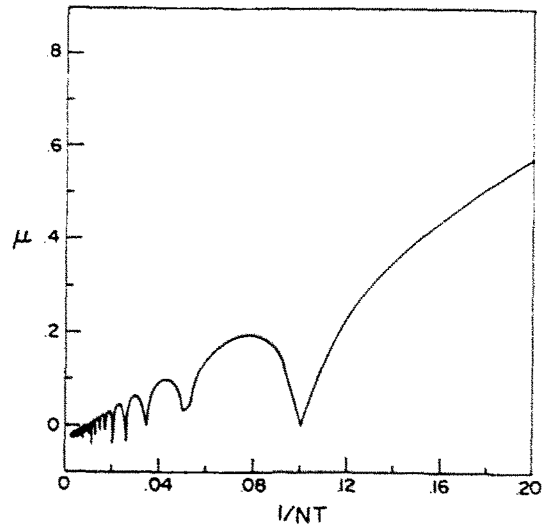

Fig. 3. Lyapunov exponents for a viscoelastic plate, $\eta = 0.0$.

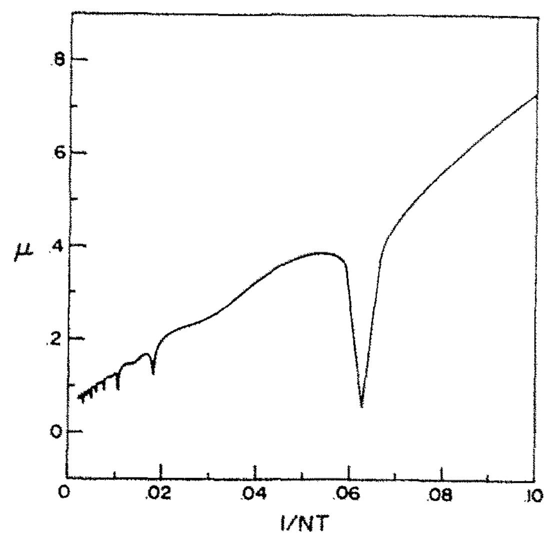

Fig. 4. Lyapunov exponents for a viscoelastic plate, $\eta = 0.4$.

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